Potential-Based Panel Methods - Variations

\[ \frac{d\theta}{dx} + \frac{\theta}{V_e} (2 + H) \frac{dV_e}{dx} = \frac{1}{2} C_f \]

\[ H = \delta^* / \theta \]

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Lecture 6
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Integral Equation for the Surface Potential

In the last lecture we arrived at an integral equation for the potential function on the surface of the configuration of interest. This expression was shown to be:

\[ \phi_P = V_\infty (x_P \cos \alpha + y_P \sin \alpha) + \int_{S_B+S_C} [(\hat{n} \cdot \nabla \phi) \phi_s - \phi (\hat{n} \cdot \nabla \phi_s)] dS \quad (1) \]

Now, the surface integral portion of this Equation can be simplified further by making a few observations. Firstly, the flow tangency boundary condition establishes that, on the surface of the body \( S_B \), \( \hat{n} \cdot V = \hat{n} \cdot \nabla \phi = 0 \). Consequently, the first term of the surface integral will be zero for the portion of the integral that includes \( S_B \).

Furthermore, the portion of the first term of the integral that includes \( S_C \) can also be eliminated using the following rationale. The velocity
component tangential to $S_C$ can be discontinuous in certain situations (a trailing vortex sheet, for example). However, the component of velocity normal to $S_C$ is continuous. Since the unit normals on both sides of $S_C$ have directly opposite directions, their contributions cancel out exactly, thus eliminating the whole first part of the integral in Equation 1 leaving us with

$$\phi_P = V_\infty (x_P \cos \alpha + y_P \sin \alpha) - \int_{S_B + S_C} \phi (\hat{n} \cdot \nabla \phi_s) dS \quad (2)$$

Hmm...according to our earlier results, we could obtain the potential of the flow by using a distribution of sources and doublets on the surface of the configuration, but we have now completely wiped off the portion relating to the source distribution, $(\hat{n} \cdot \nabla \phi) \phi_s$. What is going on? Well, we can see later (and you can read about it in Moran, pp. 282–285) that an alternative vector field $U^*$ can be used in conjunction with the divergence theorem to recover the source distribution in the expression for the potential, thus
explaining in a mathematically rigorous way the approach taken by Hess and Smith. We will not discuss this topic during lecture.

Now, the potential $\phi$ itself can be discontinuous across $S_C$ (and indeed it is in all cases of lifting flows). Using the concept of the potential jump across $S_C$ defined as:

$$\Delta \phi = \phi^+ - \phi^-$$

we can rewrite our expression for $\phi_P$ as follows

$$\phi_P = V_{\infty}(x_P \cos \alpha + y_P \sin \alpha) - \int_{S_B} \phi(\hat{n} \cdot \nabla \phi_s) dS - \int_{S_C} \Delta \phi(\hat{n} \cdot \nabla \phi_s) dS \quad (3)$$

As shown in the Figure below, $\phi^+$ is the value of the potential on the side of $S_C$ for which $\hat{n}$ goes into the fluid, while $\phi^-$ is the value of the potential on the other side. It is, of course, understood that the integral is now carried out over a single side of $S_C$. 
The potential jump $\Delta \phi$ can be easily related to the values of the potential on $S_B$. If $P_1$ and $P_2$ are the intersections of both sides of $S_C$ with $S_B$, then

$$\phi|_{P_1} \neq \phi|_{P_2}$$

Instead

$$\Delta \phi = \phi|_{P_2} - \phi|_{P_1} = \oint_{P_1}^{P_2} \nabla \phi \cdot d\mathbf{l} = \Gamma \quad (4)$$

This is not only true at the trailing edge of the airfoil, but also all along the
wake cut $S_C$. This can be easily seen by repeating the circulation calculation above for closed contour that begin and end at matching points on $S_C$. Since the value of the circulation integral is always the same, regardless of the location of the intersection of the contour with $S_C$, the potential jump $\Delta \phi = \Gamma$ everywhere on $S_C$. Please refer to the Figure below.

![Figure 2: Potential Jump Across $S_C$](image)

Finally we are left with an equation that can be used in practice for a
variety of potential-based panel methods:

\[ \phi_P = V_{\infty}(x_P \cos \alpha + y_P \sin \alpha) - \int_{S_B} \phi(\hat{n} \cdot \nabla \phi_s) dS - \Gamma \int_{S_C} \hat{n} \cdot \nabla \phi_s dS \]  

(5)

with \( \Gamma \) being related to the values of \( \phi \) on \( S_B \) according to Equation 4.

Notice, once more, that this Equation is an integral equation for \( \phi \) on \( S_B \) which can be solved either analytically (difficult) or numerically in a manner similar to that followed in the Hess-Smith method. In other words, for the numerical solution of Equation 5, we will discretize the surface of the configuration into a series of panels \([1, \ldots, N]\), we will assume a particular variation of the potential \( \phi \) on each panel (constant, linear, quadratic) and parameterize this variation with a series of coefficients (the unknowns of our equations), and we will obtain a number of independent equations equal to the number of unknowns by allowing the point \( P \) and its associated potential \( \phi_P \) to approach points on the surface of the body \( S_B \) where the
values of the potential are known according to the chosen parameterization. Again, this will result in a set of simultaneous linear equations \( Ax = b \) which can be easily solved using standard procedures.

As a reminder, the setup of panels and nodes, together with their numbering for a typical airfoil analysis problem can be found in Figure 3 below.

Figure 3: Airfoil Surface Panelization Conventions
Constant-Potential Method

This method is also known as the two-dimensional version of Morino and Kuo. In its simplest incarnation, we choose the potential to be constant on each panel of the configuration, but the values of these constants are allowed to differ from panel to panel.

\[ \phi = \phi_j \]

on panel \( j \), and, according to Equation 4,

\[ \Gamma = \phi_N - \phi_1 \]

Therefore, we have \( N \) unknowns (the values of the potential on each panel), and we need \( N \) independent equations to solve for these unknowns. These
equations can be obtained by evaluating Equation 5 at the midpoint of each panel. Defining

\[
\bar{x}_i = \frac{1}{2}(x_i + x_{i+1}) \quad \bar{y}_i = \frac{1}{2}(y_i + y_{i+1})
\]

we have

\[
\phi_i = V_\infty (\bar{x}_i \cos \alpha + \bar{y}_i \sin \alpha) - \sum_{j=1}^{N} \phi_j \int_{\text{panel } j} \mathbf{\hat{n}} \cdot \nabla \phi_s dS - (\phi_N - \phi_1) \int_{S_C} \mathbf{\hat{n}} \cdot \nabla \phi_s dS \quad \text{for } i = 1, \ldots, N
\] (6)

Once again, as we saw in the Hess-Smith method, the integrals in this equation need to be evaluated and they are most easily performed in
a coordinate system aligned with the panel that is contributing to the potential. Let \((x^*, y^*)\) be the coordinates of the midpoint of panel \(i\) on a coordinate system fixed on panel \(j\) such as the one in the Figure below.

\[
(x^*, y^*) = (\bar{x}_i, \bar{y}_i)
\]

Figure 4: Local Panel-fixed Coordinates
After all integrations are carried out, one arrives at an equation like

\[ \sum_{j=1}^{N} A_{ij} \phi_j = b_i \quad \text{for } i = 1, \ldots, N \]

which can be solved through the usual methods. See handout from Moran for details about the coefficients.

Finally, the velocity can be calculated in a variety of ways. The simplest of all is to compute the velocity at the nodes of the discretization by using

\[ V_i = \frac{\phi_i - \phi_{i-1}}{d} \quad (7) \]

The various elements of this calculation are defined in the Figure below.
Finally, with the velocities already calculated, one can continue as usual and compute the pressure distribution using Bernoulli’s equation, whose result can then be used to integrate the forces on the body.
A more accurate alternative to the constant potential method is to assume a linear variation of the value of the potential within each panel. For example,

$$\phi = \phi_j + \frac{\xi}{l_j}(\phi_{j+1} - \phi_j)$$

on panel $j$, where $\xi$ is simply the coordinate along the length of the panel (which varies from 0 to 1 from one end of the panel to the other), and where $\phi_j$ is the potential at the $j$th node. See Figure 4 for more details. Note that the potentials are now described at the nodes of the panelization and not at the midpoints of each panel. Because of this situation, we will now have a number of unknowns equal to $N + 1$, instead of $N$.

By allowing the point $P$ in Equation 5 to become each and every one of the $N + 1$ nodes of the surface panelization, we obtain a set of linear
equations for the unknowns \( \phi_j \). This has to be done quite carefully. See pp. 274-275 of Moran’s book for more details. Effectively,

\[
\phi_P = V_\infty (x_P \cos \alpha + y_P \sin \alpha)
\]

\[
+ \frac{1}{2\pi} \sum_{j=1}^{N} \left[ \phi_j + \frac{x^*}{l_j} (\phi_{j+1} - \phi_j) \right] \beta_{Pj} + \frac{y^*}{l_j} (\phi_{j+1} - \phi_j) \ln \frac{r_{Pj+1}}{r_{Pj}}
\]

\[
+ \frac{1}{2\pi} (\phi_{N+1} - \phi_1) \beta_{PN+1}
\]

with especially careful definitions of \( \beta_{Pj} \) and \( \ln \frac{r_{Pj+1}}{r_{Pj}} \). Unfortunately, the equations that result for \( \phi_P = \phi_1 \) and \( \phi_P = \phi_{N+1} \) are identical, and therefore, we need to enforce the Kutta condition explicitly to obtain a solvable system. This can be done in a manner similar to that used in the
Hess-Smith panel method:

\[
\frac{\phi_{N+1} - \phi_N}{l_N} = \frac{\phi_1 - \phi_2}{l_1}
\]

Once the linear system is solved, the tangential velocity component at the midpoints of the panels can be easily calculated as

\[
V_t(\bar{x}_i, \bar{y}_i) \approx \frac{\phi_{j+1} - \phi_j}{l_j}
\]

and the pressure distribution can be calculated with the help of the Bernoulli equation. From the pressure distribution, the force and moment coefficients can be computed in the usual way.
Equivalent Vortex Distributions

Without going into much detail, the following can be stated:

*For every doublet distribution, there corresponds an equivalent vortex distribution.*

This statement allows the rigorous mathematical development of vortex-based panel methods (such as the Hess-Smith method).

Integrals of the type

\[
\frac{1}{2\pi} \int_{\text{panel}_j} \phi \frac{y^* d\xi}{(x^* - \xi)^2 + y^*^2} = \frac{1}{2\pi} \phi \theta \bigg|_{\xi=l_j}^{\xi=l_j} - \frac{1}{2\pi} \int_0^{l_j} \frac{\partial \phi}{\partial \xi} \theta d\xi
\]
which show up in the development of doublet-based panel methods can be shown to be equivalent to a distribution of vortices of strength $\frac{\partial \phi}{\partial \xi}$ within each panel, plus two concentrated vortices at the end of each panel whose strength is equal to the value of the potential at the end of the panel.

This equivalence between doublet and vortex distributions can help shed some light into the development of vortex-based panel methods (see Moran pp. 280–282) for more details.
Source-Based Panel Methods

In our mathematical development of panel methods, we had obtained an expression for the potential at any point in the flow field, $P$, in terms of surface source and doublet distributions. The strength of the source distribution was found to be equal to the normal component of velocity at the surface. Because of the flow tangency boundary condition, this source strength turned out to be zero, and the source distribution dropped out of our equation.

However, had we defined our vector field $\mathbf{U}$ in a different way, we could have arrived at slightly different results. In particular, if the vector field of interest $\mathbf{U}^*$ is defined as follows

$$\mathbf{U}^* = \phi^* \nabla \phi_s - \phi_s \nabla \phi^*$$

where $\phi^*$ is the solution to Laplace’s equation inside the body of interest,
while \( \phi_S \) is still the potential of a unit-strength source located at a point \( P \) outside of the body of interest, we would have arrived (using both the divergence theorem and the result of the earlier definition of \( \mathbf{U} \)) at the following conclusion:

\[
\phi_P = V_\infty (x \cos \alpha + y \sin \alpha) + \int_{S_B+S_C} (\sigma \phi_s - \mu \hat{n} \cdot \nabla \phi_s) dS
\]

where

\[
\sigma = \hat{n} \cdot \nabla (\phi - \phi^*)
\]

\[
\mu = \phi - \phi^*
\]

This is the same conclusion that we had arrived at earlier, only now the source and doublet strength distributions are not as simply related to the normal component of the velocity and the potential values themselves.
They are related to the difference between $\phi$ and $\phi^*$ and its gradient, whose values at the surface of the body can be quite arbitrary. This arbitrariness can be easily exploited to make sure that the source distribution is not eliminated by the flow tangency boundary condition.

Using the equivalence between doublet and vortex distributions, the expressions obtained with this approach can be made to look very similar to the Hess-Smith method discussed earlier in class.