CME342
Parallel Methods in Numerical Analysis

Domain Decomposition Methods I

Outline:

• Overlapping methods.
• Nonoverlapping methods.
Domain Decomposition Methods

• Key Idea: Divide and Conquer

• Suitable for parallel computing.

• Overlapping Methods
  ▶ Multiplicative: Classical Alternating Schwarz
  ▶ Additive

• Nonoverlapping Methods
  ▶ Substructuring Iterative Methods

• Two Level Methods / Convergence Analysis

• Reference: Domain Decomposition, Barry Smith, Petter Bjorstad, William Gropp.
Scalability

- Ideal case 1: double # of procs, double speedup.
  - Amdahl’s law.
  - Communication cost.

- Ideal case 2: double problem size, double execution time.
  - Not possible for solving linear systems using GE since $O(n^3)$.
  - Neither for Jacobi, GS whose complexity = $O(n^2)$ nor CG = $O(n^{3/2})$.

- Scalable methods: domain decomposition, multigrid.
Model Problem: 2 Subdomains

\[ \Omega = \Omega_1 \cup \Omega_2 \]

Definitions:

- \( \Omega = \Omega_1 \cup \Omega_2 \)
- \( \Gamma_i = \partial \Omega_i \cap \Omega \) = artificial interface/boundary
- \( u^k_i = \) approx. solution on \( \bar{\Omega}_i \) after \( k \) iterations
- \( u^k_1|_{\Gamma_2} = \) restriction of \( u^k_1 \) to \( \Gamma_2 \)
- \( u^k_2|_{\Gamma_1} = \) restriction of \( u^k_2 \) to \( \Gamma_1 \)
Classical Alternating Schwarz (1869)

- Consider solving the following boundary value problem:
  \[ Lu = f \quad \text{in} \quad \Omega \]
  \[ u = g \quad \text{on} \quad \partial \Omega \]
  e.g. \( L = -\Delta \), the Laplacian operator.

- Alternating Schwarz algorithm:

  Starting with initial guess \( u_2^0 \) on \( \Omega_2 \) (actually only need values of \( u_2^0 \) on \( \Gamma_1 \)), for \( k = 1, 2, \ldots \).

  (i) Solve for \( u_1^{k+1} \) in \( \Omega_1 \):

  \[
  \begin{cases}
  L u_1^{k+1} = f & \text{in} \quad \Omega_1 \\
  u_1^{k+1} = g & \text{on} \quad \partial \Omega_1 \setminus \Gamma_1 \\
  u_1^{k+1} = u_2^k |_{\Gamma_1} & \text{on} \quad \Gamma_1
  \end{cases}
  \]
(ii) Then, solve for $u_2^{k+1}$ in $\Omega_2$:

$$\begin{cases}
Lu_2^{k+1} = f & \text{in } \Omega_2 \\
u_2^{k+1} = g & \text{on } \partial\Omega_2 \setminus \Gamma_2 \\
u_{2}^{k+1} = u_{1}^{k+1}|_{\Gamma_2} & \text{on } \Gamma_2
\end{cases}$$
Classical Alternating Schwarz (cont.)

• In each half step of alternating Schwarz:

  Solve the elliptic problem in $\Omega_i$:
  \[ Lu_i^{k+1} = f \]
  with given boundary values $g$ on the true boundary $\partial \Omega_i \setminus \Gamma_i$:
  \[ u_i^{k+1} = g \quad \partial \Omega_i \setminus \Gamma_i \]
  and latest approx. solution values on the artificial boundaries:
  \[ u_1^{k+1} = u_2^k|_{\Gamma_1}, \quad u_2^{k+1} = u_1^{k+1}|_{\Gamma_2} \]

• Note that exact solution satisfies the two subdomain problems simultaneously
  \[ \Rightarrow \text{Alternating Schwarz is consistent.} \]

• May use other boundary conditions on the artificial boundaries $\Gamma_i$.
  e.g. Robin boundary condition on $\Gamma_1$:
  \[ \alpha u_1^{k+1} + \beta \frac{\partial u_1^k}{\partial n} = \alpha u_2^k + \beta \frac{\partial u_2^k}{\partial n} \]
Example: 1D, 1 element overlap

\begin{align*}
\Omega_1 &= (x_0, x_4), \quad \Omega_2 = (x_3, x_7), \quad \Omega_3 = (x_6, x_{10}).
\end{align*}

(1) Solve for $u_{1}^{k+1}$ in $\Omega_1$:

\begin{align*}
\begin{cases}
\frac{d^2}{dx^2} u_{1}^{k+1} = f & \text{in } \Omega_1 \\
 u_{1}^{k+1}(x_0) = 0 & \text{on } \partial \Omega_1 \setminus \Gamma_1 \\
 u_{1}^{k+1}(x_4) = u_2^k(x_4) & \text{on } \Gamma_1
\end{cases}
\end{align*}

Matrix form:

\[
\begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
u_1^{k+1}(x_1) \\
u_1^{k+1}(x_2) \\
u_1^{k+1}(x_3)
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
 f_3 + u_2^k(x_4)
\end{bmatrix}
\]
Example: 1D (cont.)

• (2) Solve for $u_2^{k+1}$ in $\Omega_2$:

$$\begin{cases}
\frac{d^2}{dx^2}u_2^{k+1} = f & \text{in } \Omega_2 \\
u_2^{k+1}(x_3) = u_1^{k+1}(x_3) & \text{on } \Gamma_2 \\
u_2^{k+1}(x_7) = u_3^{k}(x_7) & \text{on } \Gamma_2 
\end{cases}$$

Matrix form:

$$\begin{bmatrix}
2 & -1 & \\
-1 & 2 & -1 \\
-1 & 2 & 
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
u_2^{k+1}(x_4) \\
u_2^{k+1}(x_5) \\
u_2^{k+1}(x_6)
\end{bmatrix} = \begin{bmatrix}
f_4 + u_1^{k+1}(x_3) \\
f_5 \\
f_6 + u_3^{k}(x_7)
\end{bmatrix}
\end{bmatrix}$$

• (3) Solve for $u_3^{k+1}$ in $\Omega_3$:

$$\begin{bmatrix}
2 & -1 & \\
-1 & 2 & -1 \\
-1 & 2 & 
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
u_3^{k+1}(x_7) \\
u_3^{k+1}(x_8) \\
u_3^{k+1}(x_9)
\end{bmatrix} = \begin{bmatrix}
f_7 + u_2^{k+1}(x_6) \\
f_8 \\
f_9
\end{bmatrix}
\end{bmatrix}$$

• $u^{k+1} = (u_1^{k+1}, u_2^{k+1}, u_3^{k+1})$. 


Alternating Schwarz vs Block GS

• Write $A$ as $3 \times 3$ block:

$$A = \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}$$

• Block GS:

$$A_{11}x_1^{k+1} = f_1 - A_{12}x_2^k - A_{13}x_3^k$$
$$= f_1 - A_{12}x_2^k$$
$$= \begin{bmatrix}
f_1 \\
f_2 \\
f_3 + x_4^k
\end{bmatrix}$$

$$A_{22}x_2^{k+1} = f_2 - A_{21}x_1^{k+1} - A_{23}x_3^k$$
$$= \begin{bmatrix}
f_4 + x_3^{k+1} \\
f_5 \\
f_6 + x_7^k
\end{bmatrix}$$

$$A_{33}x_3^{k+1} = f_3 - A_{31}x_1^{k+1} - A_{32}x_2^{k+1}$$
$$= \begin{bmatrix}
f_7 + x_6^{k+1} \\
f_8 \\
f_9
\end{bmatrix}$$
Example: 1D, 2 elements overlap

- (1) Solve for $u_1^{k+1}$ in $\Omega_1$:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{k+1}(x_1) \\ u_2^{k+1}(x_2) \\ u_3^{k+1}(x_3) \\ u_4^{k+1}(x_4) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 + u_2^k(x_5) \end{bmatrix}$$

- (2) Solve for $u_2^{k+1}$ in $\Omega_2$:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_4^{k+1}(x_4) \\ u_5^{k+1}(x_5) \\ u_6^{k+1}(x_6) \\ u_7^{k+1}(x_7) \end{bmatrix} = \begin{bmatrix} f_4 + u_1^{k+1}(x_3) \\ f_5 \\ f_6 \\ f_7 + u_3^k(x_6) \end{bmatrix}$$
Example: 1D, 2 elements overlap (cont.)

• (3) Solve for $u_{3}^{k+1}$ in $\Omega_3$:

$$
\begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
u_{3}^{k+1}(x_7) \\
u_{3}^{k+1}(x_8) \\
u_{3}^{k+1}(x_9)
\end{bmatrix}
= 
\begin{bmatrix}
f_7 + u_{2}^{k+1}(x_6) \\
f_8 \\
f_9
\end{bmatrix}
$$

• $u^{k+1} = (u_1^{k+1}(x_1), u_1^{k+1}(x_2), u_1^{k+1}(x_3), u_2^{k+1}(x_4), u_2^{k+1}(x_5),
  u_2^{k+1}(x_6), u_3^{k+1}(x_7), u_3^{k+1}(x_8), u_3^{k+1}(x_9))$

• Related to but different from block GS. Will explain later.
Example: 2D

\[ \begin{array}{|c|c|c|c|} \hline & \Gamma_1 & & \\ \hline \Omega_2 & 13 & 14 & 15 & 16 \\ \hline & 11 & 3 & 4 & 12 \\ \hline & 9 & 1 & 2 & 10 \\ \hline & 5 & 6 & 7 & 8 \\ \hline \end{array} \]

\[ \partial\Omega_1 \setminus \Gamma_1 \]

- Solve

\[ -\Delta u = f \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \]

- 1st half-step: (subscripts denote component number)

Solve for

\[ \begin{pmatrix} u_1^{k+1/2} \\ u_2^{k+1/2} \end{pmatrix}, \] given boundary conditions:

\[ \begin{cases} -\Delta u^{k+1/2} = f & \text{in } \Omega_1 \\ u^{k+1/2} = 0 & \text{on } \partial \Omega_1 \setminus \Gamma_1 \\ u_3^{k+1/2} = u_3^k, \ u_4^{k+1/2} = u_4^k & \text{on } \Gamma_1 \end{cases} \]

i.e.

\[ \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} u_1^{k+1/2} \\ u_2^{k+1/2} \end{pmatrix} = \begin{pmatrix} f_1 - u_3^k \\ f_2 - u_4^k \end{pmatrix} \]
Example: 2D (cont.)

- 2nd half-step:
  Solve for \( \begin{pmatrix} u_3^{k+1} \\ u_4^{k+1} \end{pmatrix} \), given boundary conditions:

\[
\begin{cases}
  -\Delta u^{k+1} = f & \text{in } \Omega_2 \\
  u^{k+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma_2 \\
  u_1^{k+1} = u_1^{k+1/2}, \quad u_2^{k+1} = u_2^{k+1/2} & \text{on } \Gamma_2
\end{cases}
\]

Similarly, we have

\[
\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} u_3^{k+1/2} \\ u_4^{k+1/2} \end{pmatrix} = \begin{pmatrix} f_3 - u_1^{k+1/2} \\ f_4 - u_2^{k+1/2} \end{pmatrix}
\]
Matrix Interpretation

- Assume matching grid, i.e. the discretizations of the 2 subdomains coincide in the overlapping region.

Notation:

- Write the approx. solution \( u_i \) in \( \Omega_i \) as:

\[
\begin{bmatrix}
  u_\Omega_i \\
  u_{\partial\Omega_i \setminus \Gamma_i} \\
  u_{\Gamma_i}
\end{bmatrix}
\]

- Note: \( u_{\partial\Omega_i \setminus \Gamma_i} \) are actually known; given by the Dirichlet boundary condition. They are kept as unknowns for convenience.

- Let \( A_i = \text{discrete } L \text{ on } \Omega_i \). Similar to above, it can be written as:

\[
A_i = (A_\Omega, A_{\partial\Omega_i \setminus \Gamma_i}, A_{\Gamma_i})
\]

\( A_\Omega \) = coupling between interior nodes.

\( A_{\partial\Omega_i \setminus \Gamma_i} \) = coupling between interior and true boundary nodes.

\( A_{\Gamma_i} \) = coupling between interior and artificial boundary nodes.
Example: $L = -\Delta$

- The components of $u$ in $\Omega_1$:
  
  $$u_{\Omega_1} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_{\partial \Omega_1 \setminus \Gamma_1} = \begin{pmatrix} u_3 \\ \vdots \\ u_{10} \end{pmatrix}, \quad u_{\Gamma_1} = \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix}$$

- The couplings of $A_1$ in $\Omega_1$:

  $$A_1 u_1 = (A_{\Omega_1} \ A_{\partial \Omega_1 \setminus \Gamma_1} \ A_{\Gamma_1}) \begin{pmatrix} u_{\Omega_1} \\ u_{\partial \Omega_1 \setminus \Gamma_1} \\ u_{\Gamma_1} \end{pmatrix} = \begin{pmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix}$$

- 1st row: difference eqn. at $x_1$.
- 2nd row: difference eqn. at $x_2$. 
Matrix Interpretation (cont.)

Using the new subscript notations, the alternating Schwarz algorithm can be written as the following 2 half-steps:

\[
\begin{align*}
\begin{cases}
(A_{\Omega_1} A_{\partial\Omega_1\setminus\Gamma_1} A_{\Gamma_1})
\begin{pmatrix}
  u^{k+1}_{\Omega_1} \\
  u^{k+1}_{\partial\Omega_1\setminus\Gamma_1} \\
  u^{k+1}_{\Gamma_1} \\
  u^{k+1}_{\partial\Omega_2\setminus\Gamma_2} \\
  u^{k+1}_{\Gamma_2}
\end{pmatrix}
= f_1 \text{ in } \Omega_1 \\
= g_1 \text{ on } \partial\Omega_1\setminus\Gamma_1 \\
= u^{k+1}_{\Omega_2}\mid_{\Gamma_1} \text{ on } \Gamma_1
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
(A_{\Omega_2} A_{\partial\Omega_2\setminus\Gamma_2} A_{\Gamma_2})
\begin{pmatrix}
  u^{k+1}_{\Omega_2} \\
  u^{k+1}_{\partial\Omega_2\setminus\Gamma_2} \\
  u^{k+1}_{\Gamma_2} \\
  u^{k+1}_{\partial\Omega_1\setminus\Gamma_1} \\
  u^{k+1}_{\Gamma_1}
\end{pmatrix}
= f_1 \text{ in } \Omega_2 \\
= g_1 \text{ on } \partial\Omega_2\setminus\Gamma_2 \\
= u^{k+1}_{\Omega_1}\mid_{\Gamma_2} \text{ on } \Gamma_2
\end{cases}
\end{align*}
\]

- Multiply $A_i$ and $u_i^{k+1}$ and move known quantities to the right-hand sides:

\[
\begin{align*}
(*) \begin{cases}
A_{\Omega_1} u^{k+1}_{\Omega_1} = f_1 - A_{\partial\Omega_1\setminus\Gamma_1} g_1 - A_{\Gamma_1} u^{k}_{\Omega_2}
\end{cases}
\end{align*}
\begin{align*}
(*) \begin{cases}
A_{\Omega_2} u^{k+1}_{\Omega_2} = f_2 - A_{\partial\Omega_2\setminus\Gamma_2} g_2 - A_{\Gamma_2} u^{k+1}_{\Omega_1}
\end{cases}
\end{align*}
\]
Let $\tilde{f}_i = f_i - A_{\partial \Omega_i \setminus \Gamma_i} g_i$, which is the usual right-hand side for $\Omega_i$ after eliminating unknowns on the true boundary. Then (*) becomes:

\[
\begin{align*}
A_{\Omega_1} u_{\Omega_1}^{k+1} &= \tilde{f}_1 - A_{\Gamma_1} u_{\Omega_2}^k \\
A_{\Omega_2} u_{\Omega_2}^{k+1} &= \tilde{f}_2 - A_{\Gamma_2} u_{\Omega_1}^k
\end{align*}
\]

• This is precisely block GS method applied to the linear system:

\[
\begin{pmatrix}
A_{\Omega_1} & A_{\Gamma_1} \\
A_{\Omega_2} & A_{\Gamma_2}
\end{pmatrix}
\begin{pmatrix}
u_{\Omega_1} \\
u_{\Omega_2}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{f}_1 \\
\tilde{f}_2
\end{pmatrix}
\]

• Note: The size of this linear system is bigger than $n$ since the values of $u$ on $\Omega_1 \cap \Omega_2 \neq \emptyset$ are duplicated in $u_{\Omega_1}$ and $u_{\Omega_2}$.

• Partitioning of matrix $A_\Omega$:
$A_\Omega = A_{\Omega_1} A_{\Omega_2}$
Matrix Operator Form

• Write the alternating Schwarz as 2 half-steps:

\[
\begin{align*}
    u^{k+1/2} & \leftarrow u^k + \begin{pmatrix} 0 & 0 \\ A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^k) \\
u^{k+1} & \leftarrow u^{k+1/2} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} (f - Au^{k+1/2})
\end{align*}
\]

• Define the restriction operator \( R_i \) (rectangular matrix) as:

\[
\begin{align*}
u_{\Omega_1} & = R_1 u \equiv (I \ 0) \begin{pmatrix} u_{\Omega_1} \\ u_{\Omega\setminus\Omega_1} \end{pmatrix} \\
u_{\Omega_2} & = R_2 u \equiv (0 \ I) \begin{pmatrix} u_{\Omega\setminus\Omega_2} \\ u_{\Omega_2} \end{pmatrix}
\end{align*}
\]

i.e. \( R_i u \) restricts \( u \) to the subdomain \( \Omega_i \).

• Note: \( A_{\Omega_i} = R_i A R_i^T \). Hence, alternating Schwarz can be written as:

\[
\begin{align*}
    u^{k+1/2} & \leftarrow u^k + R_1^T (R_1 A R_1^T)^{-1} R_1 (f - Au^k) \\
u^{k+1} & \leftarrow u^{k+1/2} + R_2^T (R_2 A R_2^T)^{-1} R_2 (f - Au^{k+1/2})
\end{align*}
\]

• No need to form \( R_i \) in practice; only used in analysis.
• Define $B_i = R_i^T(R_iAR_i^T)^{-1}R_i$, i.e. $B_i$ restricts the residual to subdomain $\Omega_i$, solves the sub-domain problem to generate a correction, and extends back onto the entire domain.

▷ Just like $R_i$, never form $B_i$ in practice.

• Then the alternating Schwarz iteration can be written as:

$$u^{k+1} \leftarrow u^k + (B_1 + B_2 - B_2AB_1)(f - Au^k)$$

i.e. The preconditioner $B_{MS} (\approx A^{-1})$ given by the Schwarz method is:

$$B_{MS} = B_1 + B_2 - B_2AB_1$$

• Define the error of $u^k$ by: $e^k = u - u^k$. By direct calculation,

$$e^{k+1} \leftarrow e^k - (B_1 + B_2 - B_2AB_1)Ae^k$$

$$= [I - (B_1 + B_2 - B_2AB_1)A]e^k$$

$$= (I - B_2A)(I - B_1A)e^k$$

Hence, the iteration matrix is a product of 2 matrices $\rightarrow$ multiplicative Schwarz.
Symmetrized Multiplicative Schwarz

- The preconditioner $B_{MS}$ is not symmetric, and hence cannot be used as a preconditioner for CG, even if $A$ is SPD.

- Can symmetrize the multiplicative Schwarz by adding one more intermediate step:

$$u^{k+1/3} \leftarrow u^k + B_1(f - Au^k)$$
$$u^{k+2/3} \leftarrow u^{k+1/3} + B_2(f - Au^{k+1/3})$$
$$u^{k+1} \leftarrow u^{k+2/3} + B_1(f - Au^{k+2/3})$$

i.e. Solving the subdomain $\Omega_1$ problem again.

- The preconditioner $B_{SMS}$ of the symmetric multiplicative Schwarz is:

$$B_{SMS} = B_1 + (I - B_1A)B_2(I - AB_1)$$
Additive Schwarz Method

- If we do **not** use the most updated values when solving $\Omega_2$, the Schwarz iteration becomes:

\[
\begin{align*}
    u^{k+1/2} & \leftarrow u^k + \left( A^{-1}_{\Omega_1} 0 \right) (f - Au^k) \\
    u^{k+1} & \leftarrow u^{k+1/2} + \left( 0 0 \right) (f - Au^k)
\end{align*}
\]

- Both updates can be performed simultaneously → parallelism.

- Combing the two half-steps:

\[
u^{k+1} \leftarrow u^k + \left( \left( A^{-1}_1 0 \right) + \left( 0 0 \right) \right) (f - Au^k)
\]

i.e.

\[
u^{k+1} \leftarrow u^k + (B_1 + B_2)(f - Au^k)
\]

- The preconditioner $B_{AS}$ given by the additive Schwarz is:

\[
B_{AS} = B_1 + B_2
\]

- Can be viewed as generalized block Jacobi method.
Additive vs Multiplicative

• Analogous to Jacobi vs Gauss-Seidel

• Convergence rate of additive Schwarz is slower than multiplicative Schwarz. Typically, MS requires half as many iterations as AS.

• AS is easily parallelizable. Each subdomain problem can be solved independently in parallel whereas MS solves the subdomain problem sequentially.

• Parallelize MS using multicoloring.
Numerical Example

PDE:
\[-\Delta u = xe^y \quad \text{in } \Omega\]
\[u = -xe^y \quad \text{on } \partial \Omega\]

Domain and domain partition: same as previous example.

Convergence results: iteration counts

<table>
<thead>
<tr>
<th>n</th>
<th>Overlap size</th>
<th>MS</th>
<th>AS</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>4</td>
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<tr>
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</tr>
<tr>
<td>31</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

- Both AS and MS are insensitive to overlap size.
- About a factor of 2 difference between the number of iterations for MS and AS methods.
Practical Considerations

• For relatively large subdomain cases, exact subdomain solves may be too expensive to carry out.

• Since the Schwarz method is typically used as preconditioner for a Krylov subsp. method, we may solve the subdomain problems approximately, i.e. $A_{\Omega_i}^{-1} \approx \tilde{A}_{\Omega_i}^{-1}$.

• Krylov subsp theory requires that $\tilde{A}_{\Omega_1}^{-1}$ is a linear operator, e.g. a fixed number of sweeps of GS, or PCG solved to machine precision.

• Several steps of a Krylov subsp method is not a linear operator.

• However, experience indicates that if the local problems are solved accurately enough, convergence of the outer iteration is not affected much even if the local solver is a non-linear operator such as a Krylov subsp method.