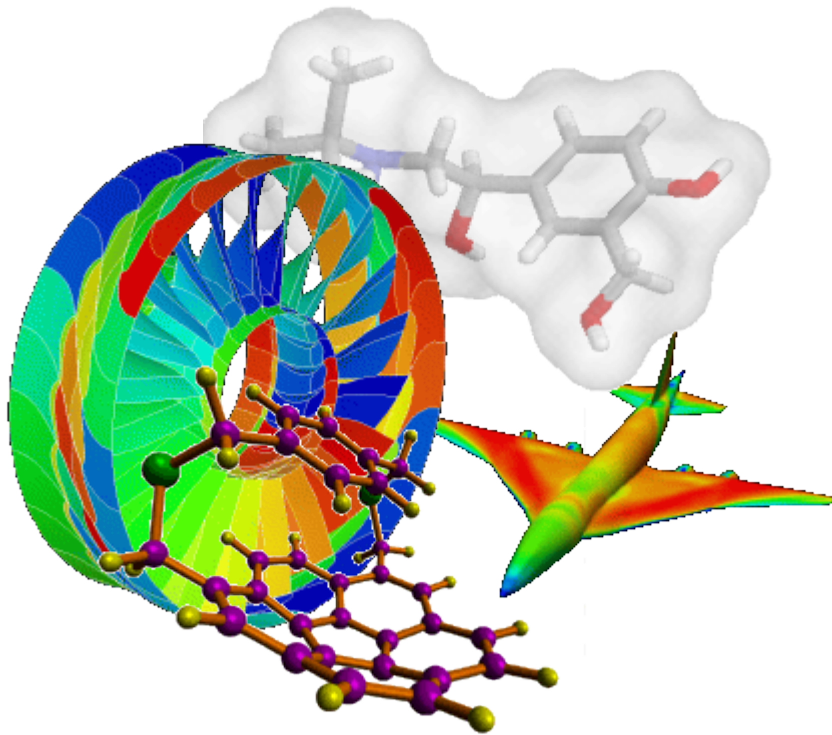
CME342/AA220/CS238 - Parallel Methods in Numerical Analysis

Fast Fourier Transform



June 4, 2014

Lecture 17

Discrete Fourier Transform

- Let i=sqrt(-1) and index matrices and vectors from 0.
- The DFT of a vector x of dimension n is:

$$y_k = \sum_{j=0}^{n-1} \omega_n^{kj} x_j$$

In matrix form y=Fx,where F is the n*n matrix defined as:

$$F[j,k] = \varpi_n^{(j*k)}$$

ϖ_n is:

$$\varpi_n = e^{(-2\pi i/n)} = \cos(2\pi/n) - i*\sin(2\pi/n)$$

- ϖ_n is the nth root of unity: $(\varpi_n)^n=1$

Fast Fourier Transform

- Generally attributed to Cooley and Turkey (1965), but FFT algorithm in Gauss notes (1805)
- Several different algorithms available:
  - Decimation in time
  - Decimation in frequency
  - Prime factor algorithm
  - Bluestein approach
  - …………
- It reduces the computational complexity from O(n^2) to O(n log n). For n=10^6, if FFT=1sec, DFT=24h!!

References:

Van Loan "Computational Frameworks for the FFT", SIAM

Briggs and Henson " The DFT", SIAM

Discrete Fourier Transform

$$F_1 = \begin{bmatrix} 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & \omega_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

ω is called twiddle factor

Fast Fourier Transform

How to compute the DFT in O(n log n) operations?
Establish a connection between F(n) and F(n/2). Repetition of this process is the heart of the radix-2 fft

$$\Pi_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \qquad F_4 \cdot \Pi_4 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{array}\right]$$

Defining $\Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = diag(1,\omega_4)$ and using $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$F_4 \cdot \Pi_4 = \begin{bmatrix} F_2 & \Omega_2 F_2 \\ F_2 & -\Omega_2 F_2 \end{bmatrix}$$

Radix-2 splitting

When n=2m

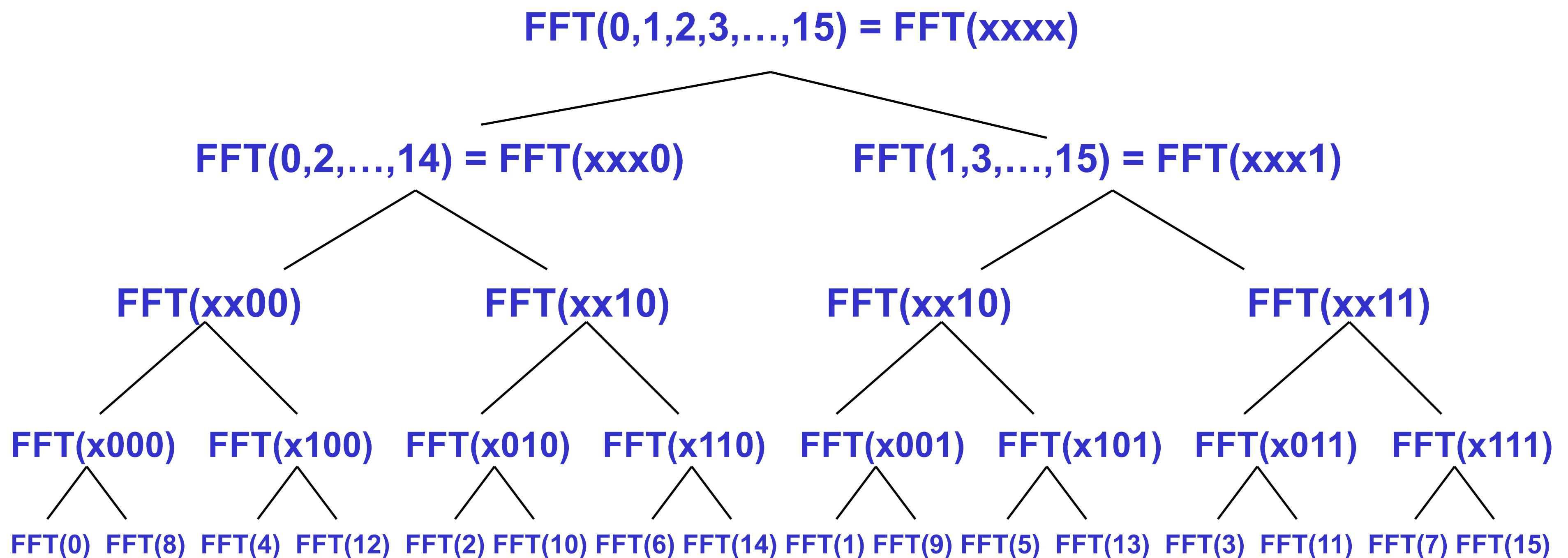
$\Omega_m = \mathrm{diag}(1, \omega_n, \ldots, \omega_n^{m-1})$

$$F_n \cdot \Pi_n = \begin{bmatrix} F_m & \Omega_m F_m \\ F_m & -\Omega_m F_m \end{bmatrix} = \begin{bmatrix} I_m & \Omega_m \\ I_m & -\Omega_m \end{bmatrix} (I_2 \otimes F_m)$$

The splitting can be generalized to n=pm (see Van Loan)

Several different algorithms can be recast in this framework

An iterative method

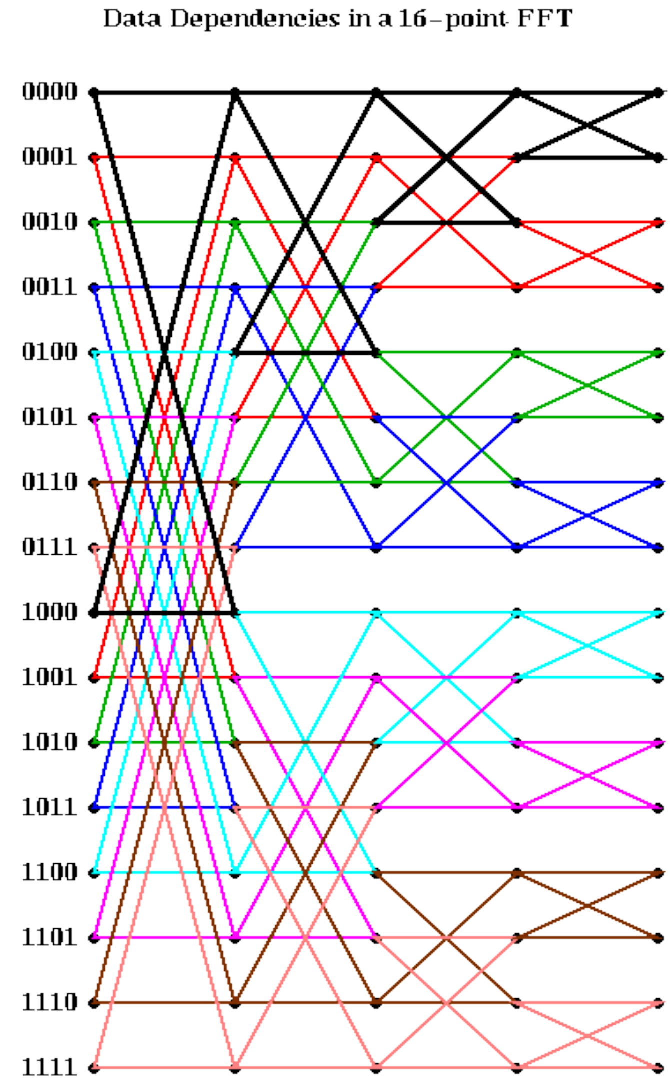


- The call tree of the d&c FFT algorithm is a complete binary tree of log m levels
- Practical algorithms are iterative, going across each level in the tree starting at the bottom (at the leaves level, we have scalar 1-point DFT $F_1 x_k = x_k$)
- Algorithm overwrites v[i] by (F*v)[bitreverse(i)]

Data dependencies in FFT

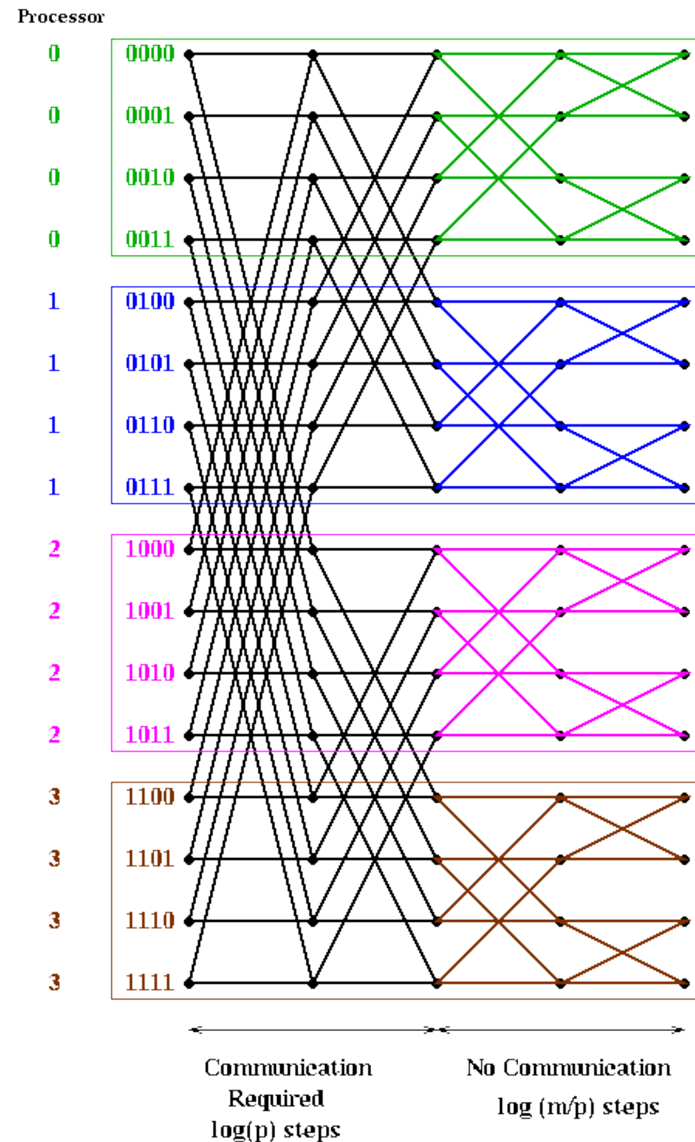
Data dependencies in 1D FFT:

- Butterfly pattern



Block layout for FFT

- Using a block layout:
  - (m/p) contiguous elements per processor
- No communication in the last log m/p steps
- Each step requires fine-grained communications in first log p steps

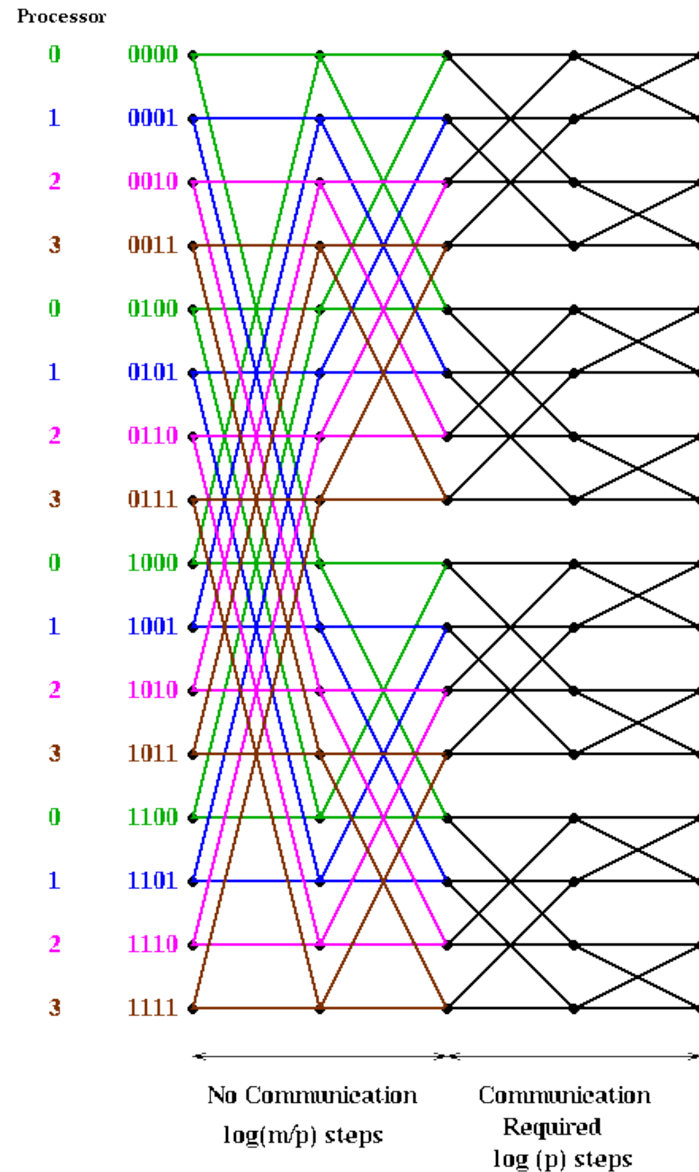


Cyclic layout for FFT

•Using a cyclic layout:
 1 element per processor,wrapped

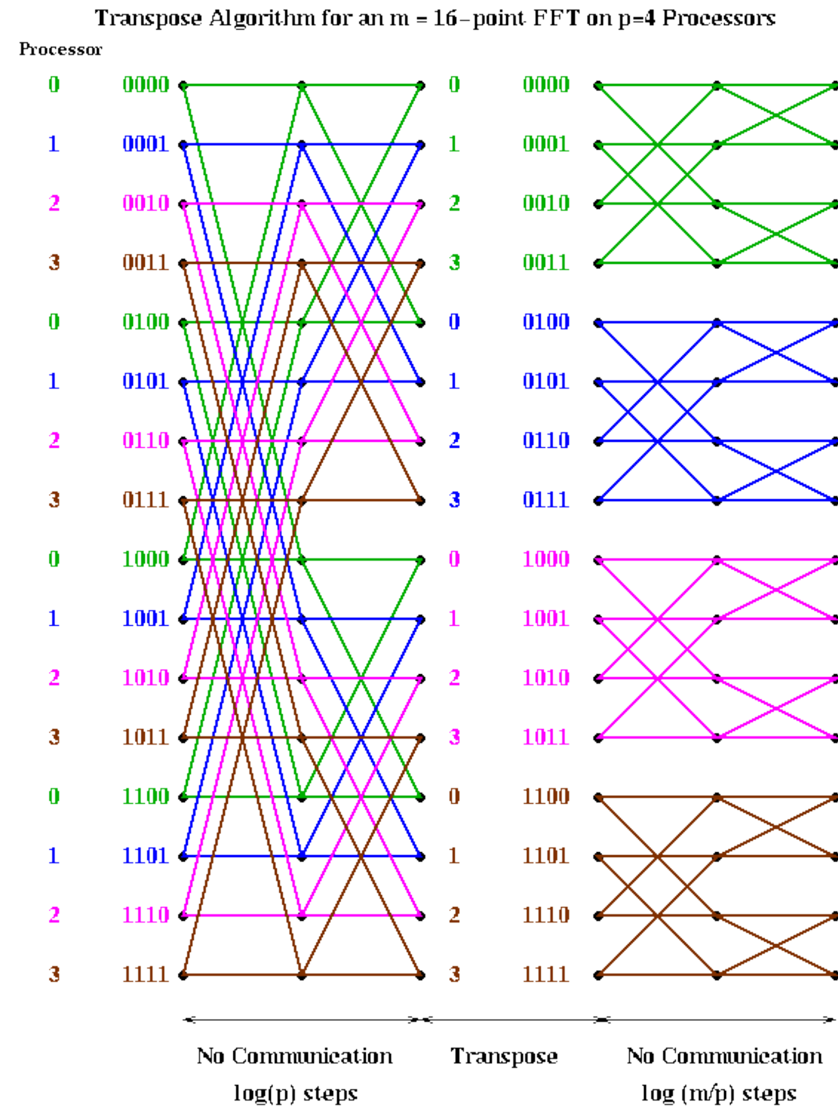
•No communication in the first log m/ p steps

•Communication in last log p steps



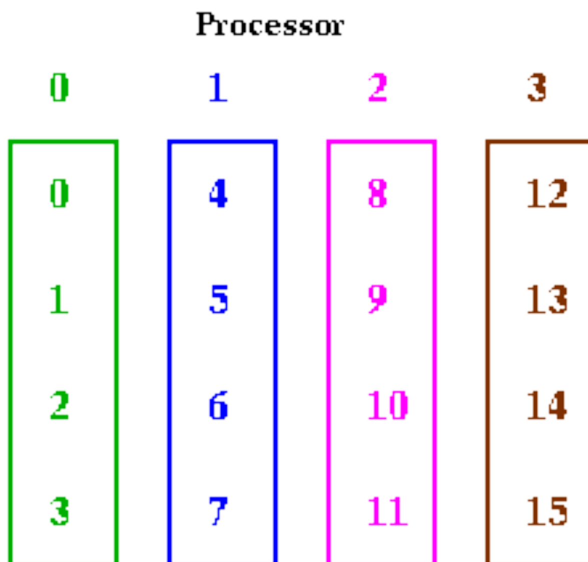
FFT with transpose

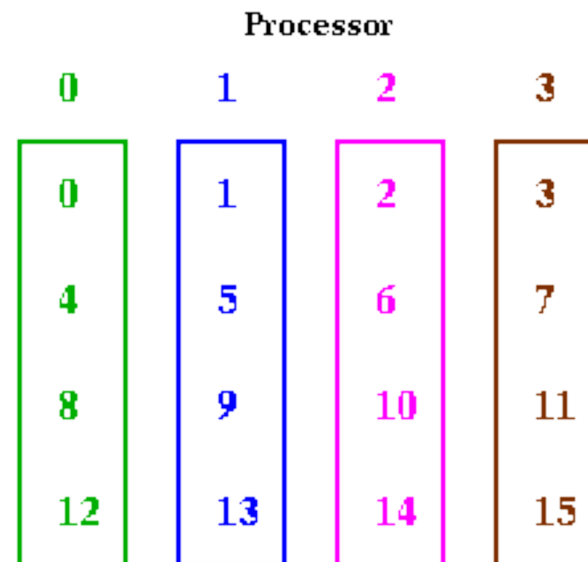
- If we start with a cyclic layout for first log(p) steps, there is no communication
- Then transpose the vector for last log(m/p) steps
- All communication is in the transpose



FFT with transpose

- Analogous to transposing an array
- View as a 2D array of n/p by p





Higher dimension FFT

- FFTs on 2 or 3 dimensions are define as 1D FFTs on vectors in all dimensions.
  E.g., a 2D FFT does 1D FFTs on all rows and then all columns

- There are 3 obvious possibilities for the 2D FFT:
  (1) 2D blocked layout for matrix, using 1D algorithms for each row and column
  (2) Block row layout for matrix, using serial 1D FFTs on rows, followed by a transpose, then more serial 1D FFTs
  (3) Block row layout for matrix, using serial 1D FFTs on rows, followed by parallel 1D FFTs on columns

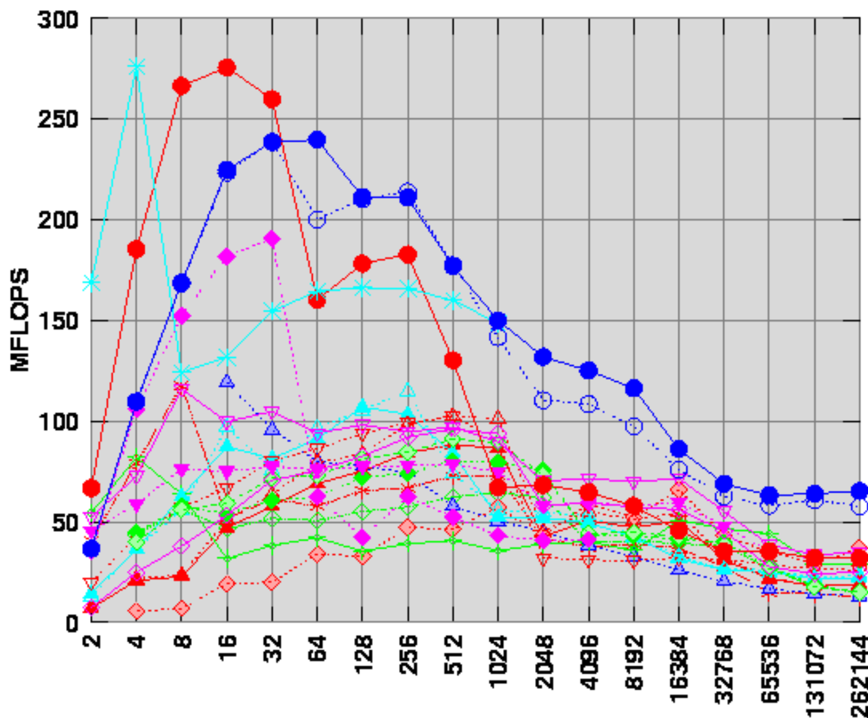
- For a 3D FFT the options are similar
  2 phases done with serial FFTs, followed by a transpose for 3rd
  can overlap communication with 2nd phase in practice

FFT libraries

•**Do not write your own fft library**, use available vendor or freeware libraries

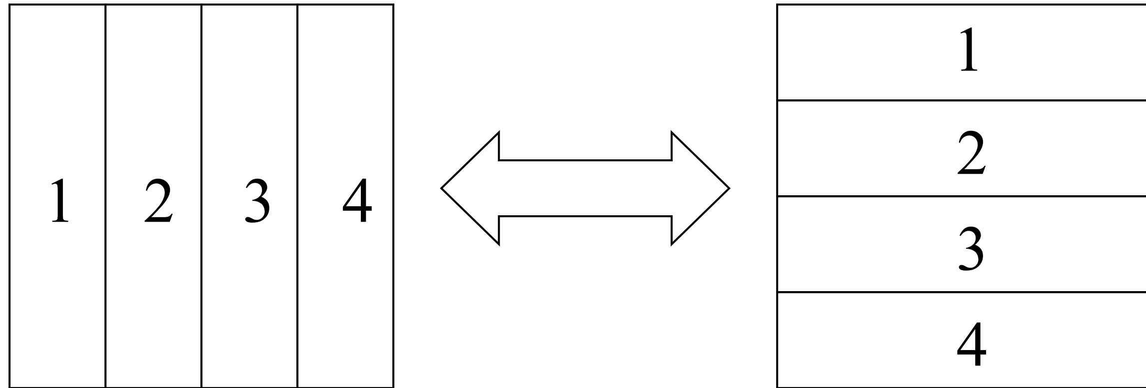
•FFTW is a fast implementation (both serial and parallel):

- •Fast (similar concept to ATLAS, autotuning)
- •Callable from C and Fortran
- •Parallel transforms use Cilk on SMP and MPI on distributed memory



//www.fftw.org

Direct exchange transpose

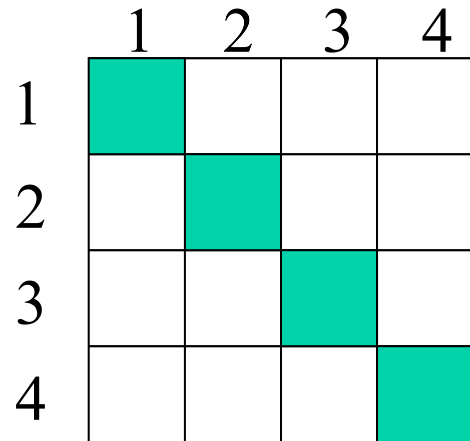


Implement the transpose in N-1 steps:

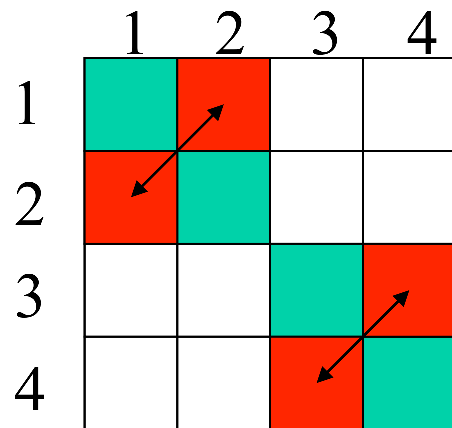
- At the ith step, the processor j exchange data with the processor number XOR(j-1,step)
- XOR(k,l) is the logical exclusive OR operation applied to the binary representation of the integer k and l.

```
np=4
do istep=1,np-1
 do j=1,np
   idest=xor(j-1,istep)
    print *,"Step:",istep," Processor:",j,"Destination",idest
end do
end do
```

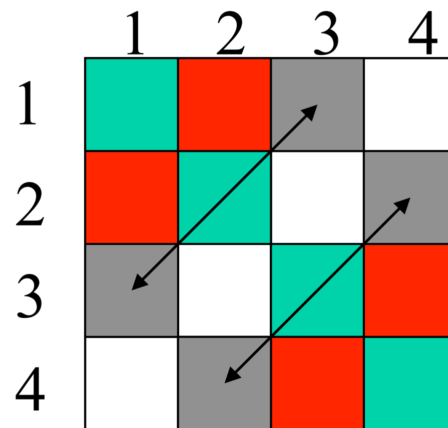
Direct exchange transpose



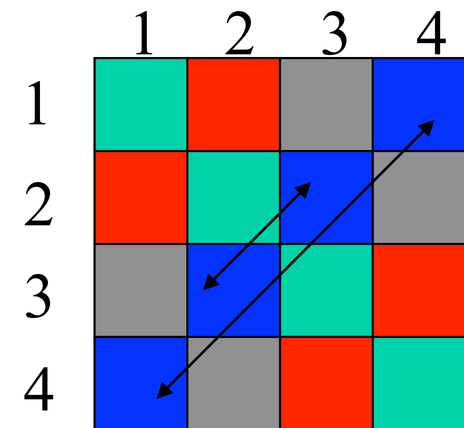
	Proc 1	Proc 2	Proc 3	Proc 4
Step 1	2	1	4	3
Step 2	3	4	1	2
Step 3	4	3	2	1



Step 1



Step 2



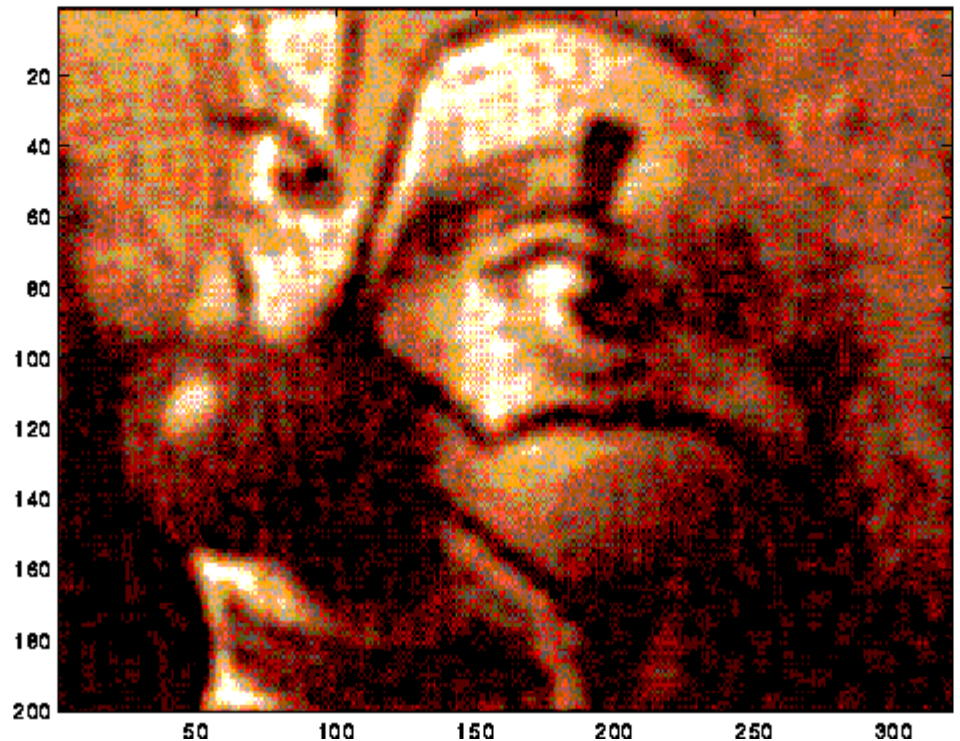
Step 3

Application of FFT

- Numerical integration
- Spectral methods for the solution of PDE
- Fast Poisson solvers
- Image processing
- Digital filtering

Image compression

Image = 200x320 matrix of values
Compress by keeping largest 2.5% of FFT components



Fast Poisson Solver

$$\nabla^2 \phi = f$$

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = f$$

$$-k_x^2 \tilde{\phi} + \frac{\partial^2}{\partial y^2}\tilde{\phi} = \tilde{f}$$

$$-k_x^2 \bar{\tilde{\phi}} - k_y^2 \bar{\tilde{\phi}} = \bar{\tilde{f}} \Rightarrow \bar{\tilde{\phi}} = \frac{\bar{\tilde{f}}}{-\left(k_x^2 + k_y^2\right)}$$

Fast Poisson Solver

- Compute Fourier coefficient f_{ik} of right hand side
  - Apply 2D FFT to values of f(i,k) on grid
- Compute Fourier coefficients ϕ_{ik} of solution
  - Divide each transformed f(i,k) by function of wave number (i,k)
- Compute solution $\phi(x,y)$ from Fourier coefficients
  - Apply 2D inverse FFT to values of f(i,k)

You can apply FFT in one direction and use FD in the other